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Scaling and river networks: A Landau theory for erosion

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We propose a coarse-grained theory for the formation of a river network in the form of a Langevin equation for the erosion of the landscape coupled to a conservation law for the surface water flow. We claim that this is the universal form for large-scale behavior. We show by simulations of a discrete model that represents the same dynamics that the slope-area law, the basin size distribution law, and Horton's laws agree with real rivers. We discuss the relationship to optimal channel networks and to self-organized criticality. [S1063-651X(97)50207-8]

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Natural river networks have attracted a good deal of attention in the physics and geophysics communities, and a large number of models have appeared that attempt to give an explanation for the remarkable statistical properties of these systems [1–4]. The spirit of much of this work is to try to abstract from the details of the geological processes a simple description that will account for the large-scale, coarse-grained properties of the network. In this paper we present a model of this type. Our model is similar to that of Inaoka and Takayasu [2] and of Sinclair and Ball [3] but also has significant differences. Our theory is intended to serve as a unified model of erosion and is based on a continuum formulation that we believe to capture the important features that survive on coarse graining. If we are correct, much of the previous work will have the same large-scale properties as what we present here.

The remarkable statistical properties of river basins have been known for some time [5,6]. We will focus on a few of the laws that we consider to be central, and that we have verified for the model to be presented. The most important of these is the slope-area law, which was derived from field observations [7]: the slope of the river bed s scales with a power of the basin area Q:

$$s \sim Q^{-\theta}, \tag{1}$$

where the value of the exponent $\theta \approx 0.5$ has been carefully measured [7]. The distribution of the drainage area also obeys a power law: $P \sim Q^{-\beta}$, where P is the fraction of the landscape for which the drainage area is larger than a given value Q. The value of the exponent is $\beta \approx 0.43$ [4,8].

The best known of the statistical properties are Horton's laws [9], which are relations between the number and length of different parts of the network. They say, in effect, that the streams form a random branching fractal. Consider the Strahler scheme for ordering the streams (i.e., up ends of the streams are order 1; when two or more streams of the same order join, the order increases by one; when streams of different order join, the higher stream order prevails.) Let N_ω denote the number of streams of order ω, and L_ω their averaged length. Horton's laws state that the branching ratio $R_B = N_\omega / N_{\omega+1}$ and the length ratio $R_L = L_{\omega+1}/L_\omega$ are independent of ω. The fractal dimension [10] of the network is given by $d_c \ln(R_B)/\ln(R_L)$, where d_c is the fractal dimension of the individual streams [11]. For many networks the values $R_B \approx 4, R_L \approx 2$ are found [9] along with $d_c \approx 1.1$ — 1.2 [11,12]. Our model will turn out to obey all these laws.

We start with the observation that landscapes seem to have scale invariance [13]: they are close to being self-affine fractals. This means that if we consider a topographic map and rescale the coordinates $\mathbf{r}$ on the map so that $\mathbf{r} \mapsto b\mathbf{r}$, and the height differences by $\Delta h \mapsto b^\alpha \Delta h$, where $\alpha < 1$ we get a statistically identical landscape. Since erosion by rivers are among the processes that form landscapes, the scale invariant statistical properties of mature river networks should have a close connection with the scale invariance of the landscape.

Now let us focus on the erosion process, and make some simplifying assumptions (which could be easily modified): we assume that the only source of water is from a uniform rainfall and neglect underground flows. The land is geologically uniform and initially structureless. We also assume that the material washed away by the river is carried entirely to the sea, and is not redeposited. This is the limit of slow erosion and fast flows.

To formulate the coarse-grained erosion law we use an argument that is standard in the theory of random rough surfaces [14] and that, in turn, is based on the classic work of Landau and Ginzburg [15]. We note first that the absolute height of the landscape should play no role in the local erosion. Thus we write

$$\partial h / \partial t = F(\nabla h, \nabla^2 h, |\nabla h|^2, \ldots) + \eta(\mathbf{r}, t), \tag{2}$$

where $\eta(\mathbf{r}, t)$ is a noise term that accounts for small scale random processes.

Further, we argue that the functional F is analytic in the gradients: it is the result of averaging over local fluctuating processes [16]. Now we are interested in large-scale statistical properties. When we rescale a self-affine surface the gradients *decrease*. Thus we should be able to expand F in a power series:

$$F = A + \mathbf{B} \cdot \nabla h + C|\nabla h|^2 + D\nabla^2 h + \cdots \tag{3}$$

We can interpret these terms. The first is a uniform change in height that might correspond to geological uplift. For our case we can set $A = 0$. The second term involves a

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vector **B**, which gives a preferred direction of flow. Since local flows have no preferred direction (except down) we must set $\mathbf{B}=\mathbf{0}$. The third term corresponds to erosion proportional to s^2, the squared slope. This sort of law has been considered in the literature [17] along with others. It has a special significance since it is the lowest-order term, and thus the dominant one when we rescale. The last one that we keep can be thought of as sedimentation and smoothing: it rounds hilltops and fills valleys [18]. The equation for the landform is

$$\partial h/\partial t = C|\nabla h|^2 + D\nabla^2 h + \cdots + \eta(\mathbf{r},t). \tag{4}$$

This is the Kardar-Parisi-Zhang (KPZ) equation [19], which has been extensively studied. There has been a previous application of this equation to river networks [20]. In this form it is clear that the equation can generate self-affine landscapes. The higher-order terms represented by the dots are *irrelevant* in the sense that they disappear upon rescaling.

The other ingredient in our theory is the water. We define **q** as the flux of water per unit width of landscape. Our assumptions (uniform rainfall and no ground water) imply that $q \propto Q$, where Q is the basin area. The vector **q** satisfies the following:

$$\boldsymbol{\nabla}\cdot\mathbf{q} = R, \tag{5}$$

where R is the rainfall per unit area. Further, water runs downhill. Thus,

$$\hat{\mathbf{q}} \equiv \mathbf{q}/q \propto -\nabla h. \tag{6}$$

Finally, we insist that there is no erosion in the absence of water. That means that the coefficient C of the erosion term must be a function of q, which vanishes as $q \to 0$. There is no particular reason why C should be analytic, so we propose on the basis of simplicity an erosion rate linear in the flow: $C = -cq$. Putting this all together we get

$$\partial h/\partial t = -cq|\boldsymbol{\nabla} h|^2 + D\nabla^2 h + \eta(\mathbf{r},t). \tag{7}$$

Equations (5)–(7) constitute our Landau theory.

Formulations similar to this one have been proposed before. A theory of this type was given by Smith and Bretherton [21] some time ago, and discussed by Tarboton *et al.* [6] in the context of stream initiation. Our equations differ from theirs in that they conserve sediment so that the right-hand side of Eq. (7) is of the form $-\boldsymbol{\nabla}\cdot[\hat{\mathbf{q}} q^m s^n]$. Our Eq. (7) corresponds (up to an irrelevant term) to $m=n=2$. The recent work of Sinclair and Ball [3] proposes a set of equations like ours with a term of the form $q^a s^b$ of which our equation is a special case. (As we will see, our solution to these equations is quite different from that of Ref. [3].)

Because the landform generated by Eq. (7) is coupled to the water flow (which changes with the landscape) the solutions to the coupled set are quite unlike those of the ordinary KPZ equation. With suitable boundary conditions, the landscape will approach a dynamic steady state where the river network and the landform do not change. This steady state is a feature of many of the models that have been proposed. It corresponds to the simple statement that large rivers are long lived [22].

To understand the steady state we use the approach of Smith and Bretherton [21] who point out that an obvious kind of steady state is one in which the erosion is uniform everywhere. If we neglect smoothing and noise (as we will do from this point on), we can write

$$\partial h/\partial t = \text{const} = -cq|\boldsymbol{\nabla} h|^2, \tag{8}$$

which amounts to having $s \propto 1/q^{1/2}$, that is, exactly the slope-area law of Eq. (1). If this state is attained it will have the observed slope-area law in a natural way, and is certainly stationary [23]. It remains to show that featureless landscapes tend towards this state, and that it is stable. To investigate this question we turn to numerical solutions of a discrete model, which is an approximate realization of our set of equations.

Our discrete model is very similar to that of Refs. [2,3] (though our boundary conditions are not). We consider a triangular lattice of mesh points that represents our landscape. Every point has two variables: the height h and the flow q. The water flows on the bonds of the lattice, and every node has one outflowing bond, the one that is the steepest. At every time step (doing parallel updates) the drainage area is calculated from the landscape, and the height is decreased according to the erosion rule $\Delta h = -|\boldsymbol{\nabla} h|^2 q \Delta t$. The gradient is measured on the outflowing edge. If there are no lakes in the initial height distribution (no nodes with all neighbors higher than itself), then using sufficiently small Δt, no lakes are created. Thus we were able to ignore the special treatment of lakes, which are generally present only in the initial stages of the erosion process, and do not affect the stationary state.

Initially the landscape is a hillside with a little noise: $h(x,y,t=0) = s_0(y + dy\,\mathfrak{R}(x,y))$, where y is the north-south coordinate, s_0 is the initial slope of the hillside, $\mathfrak{R}(\,)$ is uniform random number from [0,1], and dy is the lattice constant. These initial conditions ensure the absence of lakes. The boundary conditions are periodic in the east-west direction, infinite wall on the north side (this is the upper end of the hillside), and outflowing on the south side. The slope of the outflowing edges on the outflowing side are taken to be fixed. With these boundary conditions the stationary state is such that the whole landscape erodes with the same rate everywhere. We can think of this as representing a plateau that has been upthrust and that starts to erode. This boundary condition is in contrast with fixed *height* at the outflowing edge used by other authors [2,3]: in that case the stationary state occurs when nearly all of the material has been washed away and a different slope-area law holds [3].

In our simulations we find that the initial stages of river formation corresponds to rivers valleys that start at the bottom edge and elongate, compete, and eventually reach a stationary state with one large river. Figure 1 depicts a typical stationary river network. Taking the lattice constant to be unit length, the slopes at the outflowing edge also one, and measuring the discharge as the number of the nodes in the basin area, the rivers reach the stationary state at around unit time. The corresponding landscape is shown on Fig. 2.

The following statistical results were obtained by averaging 20 independent simulations of size 256×256. As expected, the slope-area law (Fig. 3) holds with exponent $1/2 \pm 10^{-6}$. The great accuracy is understandable if we accept that the slope-area law is an attractive fixed point of the dynamics: if any node does not satisfy the law, it will erode

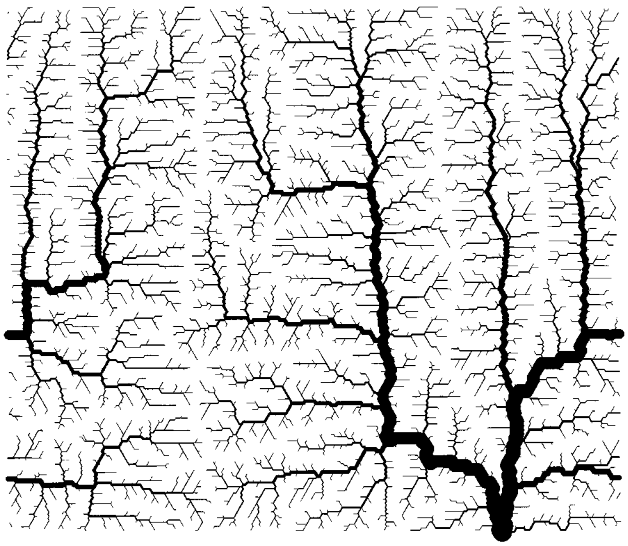

FIG. 1. A typical stationary river network on a 256×256 triangular lattice. For better visualization, the stream is drawn with linewidth proportional to the square root of the discharge. Only streams with discharge $q \geqslant 10$ are displayed.

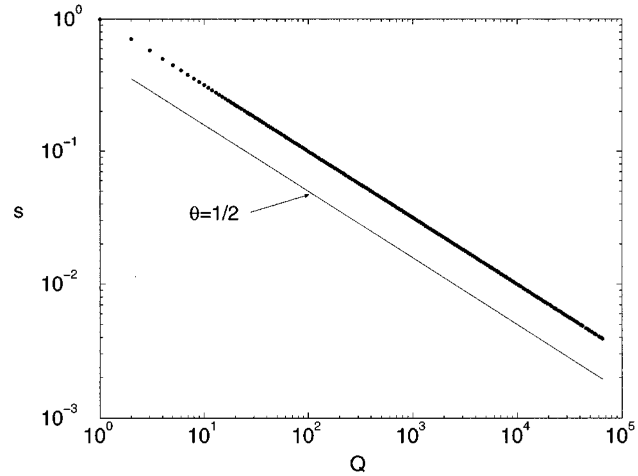

FIG. 3. The slope-area law obtained by the simulation. The exponent is $\theta = 1/2 \pm 10^{-6}$. The great accuracy is the consequence of the attractive nature of the fixed point of the dynamics.

faster or slower than its neighbors towards a height that satisfies the law. The cumulative distribution of the basin area is depicted on Fig. 4. The value of the exponent is $\beta = 0.45 \pm 0.02$. Horton's laws are shown on Fig. 5, the branching ratio is $R_B = 4.0 \pm 0.2$, the length ratio is $R_L = 2.3 \pm 0.1$. The dimension of the individual streams d_c is measured [12] from the scaling of the average river length with the system size: $\langle l_i \rangle \sim L^{d_c}$ (where l_i is the distance of site i from the root on the network). Using $L = 64$, 128, and 256, we obtained $d_c = 1.05$, giving network fractal dimension 1.85 ± 0.15. This value of the fractal dimension is somewhat lower than the expected 2 for space filling networks. The probable explanation is the low value of d_c: in our hillside initial conditions the rivers are ''stretched'' in north-south direction, making them more linear (d_c closer to 1).

There is another approach [4,24] to the problem of river networks that appears quite different from ours, namely, the idea that rivers are *optimal channel networks* (OCN's): connected branching patterns that minimize a functional that represents dissipation. It is well known that for systems far from equilibrium no functional exists in general that gives the dynamics in the usual sense that $\partial h / \partial t = \delta \mathcal{F} / \delta h$. If there were such a functional we could understand OCN's by noting that $\partial h / \partial t = 0$, the stationary state, would occur if $\mathcal{F}$ is at a minimum. However, our equations are not of this form.

The solution to this quandry was given by Sinclair and Ball [3] who point out that a functional can exist that gives the stationary state, but not the complete dynamics. It is easy to see that the height function h and flow $\mathbf{q}$ that minimize

$$\mathcal{F}[h, \mathbf{q}] = \int \{ h(\boldsymbol{\nabla} \cdot \mathbf{q} - R) + q^{1/2} \} d^2 x \qquad (9)$$

obey both Eq. (5) and Eq. (1). However this variational principle does not produce the dynamics [Eq. (7)]. There is no free energy that would produce the dynamics of the initial stages of the erosion.

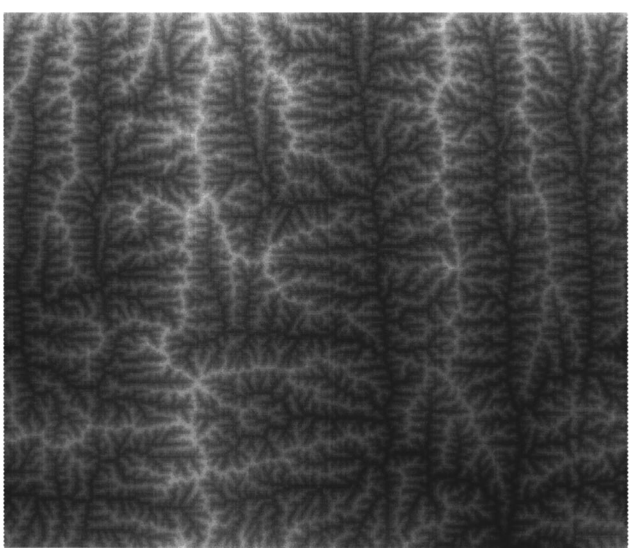

FIG. 2. The landscape created by the river of Fig. 1. The gray scale is proportional to height, with white corresponding to high.

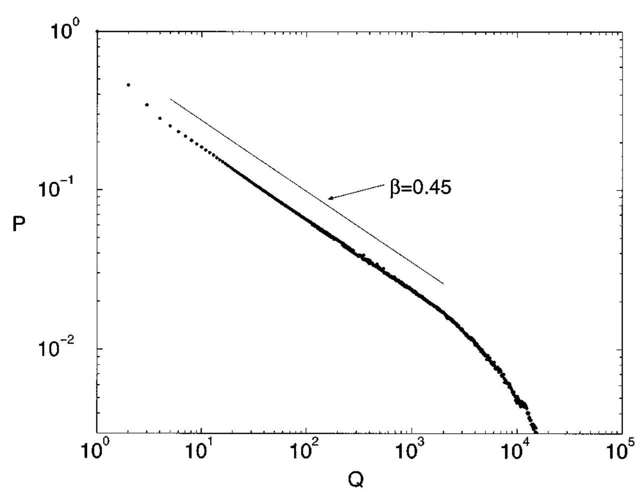

FIG. 4. The cumulative basin area distribution $P(Q)$ (the fraction of the landscape for which the drainage area is larger than a given Q). The value of the exponent, $\beta = 0.45 \pm 0.02$ agrees with Ref. [4].

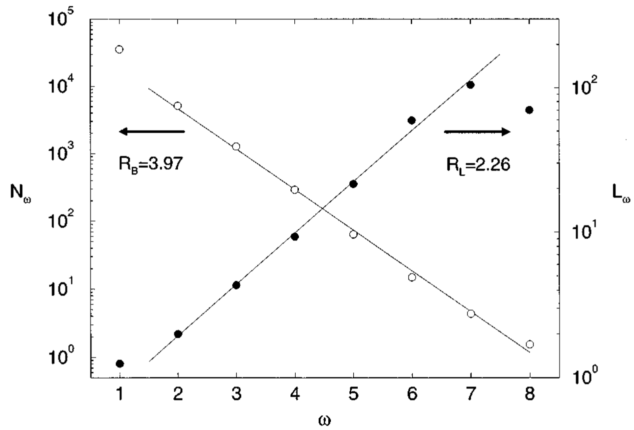

FIG. 5. Horton's laws for the branching ratio (○): $R_B=4.0\pm0.2$, and the length ratio (•): $R_L=2.3\pm0.1$. With stream dimension $d_c=1.05$, the fractal dimension of the network is $D=d_c\ln(R_B)/\ln(R_L)=1.85\pm0.15$, somewhat lower than the expected space filling 2.

In the erosion process there are sudden large-scale events that have some similarity with the avalanches of self-organized critical (SOC) systems [25]. In fact, there is a formulation of SOC dynamics [26] that resembles ours in that it involves a Langevin-like equation whose parameters are a dynamical variable (cf. $C=-cq$). However, our theory does not represent SOC processes, though there are similarities. The events in our system that are most like avalanches are river basin capture: when part of the basin area gets connected to another river. These change a macroscopic part of the flow pattern and are fast and nonlocal like avalanches, and they are essential during the evolution of the river network. But they completely disappear from the stationary state, and are not dominant for the formation of the large-scale structures. In SOC the avalanches are the only means to transmit information between the different parts of the system, and dominate any large-scale structure. In our case it is the river network itself, while eroding slowly, which transmits information.

In summary, our treatment of river networks differs from earlier work in that it emphasizes the properties of the dynamics which should survive coarse graining. We make a strong claim, that the dynamics given by Eqs. (5)–(7) is a universal theory for the large-scale structure. We have shown that, at least, there is a reasonably satisfactory agreement with the empirical satistical laws that are gleaned from field observations of real rivers. We hope that generalizations of our work to allow ground water, storms, etc., could shed some light on how these processes affect landscapes, and could even, in the best case, give useful information on, for example, the statistics of floods.

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